

Local formulae for combinatorial Pontrjagin classes

Alexander A. Gaifullin

Abstract

By $p(|K|)$ denote the characteristic class of a combinatorial manifold K given by the polynomial p in Pontrjagin classes of K . We prove that for any polynomial p there exists a function taking each combinatorial manifold K to a *rational simplicial cycle* $z(K)$ such that: (1) the Poincaré dual of $z(K)$ represents the cohomology class $p(|K|)$; (2) a coefficient of each simplex Δ in the cycle $z(K)$ is determined only by the combinatorial type of $\text{Lk } \Delta$. We also prove that if a function z satisfies the condition (2), then this function automatically satisfies the condition (1) for some polynomial p . We describe explicitly all such functions z for the first Pontrjagin class. We obtain estimates for denominators of coefficients of simplices in the cycles $z(K)$.

1 Introduction

The following general problem was studied by many researchers, see [3, 4, 5, 8, 13]. Given a triangulation of a manifold one need to construct a cycle, whose Poincaré dual represents a given Pontrjagin class of this manifold. In addition, one usually wants the coefficient of each simplex in this cycle to be determined by the structure of the manifold in some neighborhood of the simplex. First let us discuss the most important results concerning this problem.

A. M. Gabrielov, I. M. Gelfand and M. V. Losik [4, 5] found an explicit formula for the first rational Pontrjagin class of a smooth manifold. To apply this formula one need the manifold to be endowed with a smooth triangulation satisfying some special condition. In their paper [8], I. M. Gelfand and R. D. MacPherson considered simplicial manifolds endowed with some additional combinatorial data called a *fixing cycle*. A fixing cycle is a combinatorial analogue of a smooth structure and can be induced by a given smooth structure. For simplicial manifolds with a given fixing cycle I. M. Gelfand and R. D. MacPherson constructed rational cycles whose Poincaré dual represent the normal Pontrjagin classes of the manifolds. In these cycles the coefficient of simplex depends both on the combinatorial structure of the neighborhood of this simplex and on the restriction of the fixing cycle to this neighborhood.

The other approach is Cheeger's. In [3] he obtained explicit formulae for cycles whose Poincaré dual represent *real* Pontrjagin classes. In these cycles the coefficient of simplex depends only on the combinatorial type of the link of this simplex. Cheeger's formula can be applied for any pseudomanifold. It is unknown if the obtained cycles are rational.

Assume that for any combinatorial manifold K we have a *cycle* in the oriented simplicial chains of K given by

$$z(K) = \sum_{\Delta \in K, \dim \Delta = \dim K - n} f(\text{Lk } \Delta) \Delta$$

where the value $f(L)$ is determined by the isomorphism class of the oriented $(n-1)$ -dimensional PL sphere L and the function f does not depend on the manifold K . Then we say that z is a *characteristic local cycle* of codimension n . The function f is called a *local formula* for this characteristic local cycle. We prove that for any rational characteristic class $p \in H^*(BPL; \mathbb{Q})$ there exists a *rational* characteristic local cycle z such that the Poincaré dual of the cycle $z(K)$ represents the cohomology class $p(|K|)$ for any combinatorial manifold K . (BPL is the classifying space for stable PL bundles.) This improves a theorem of N. Levitt and C. Rourke [13]. They obtained a similar result for the cycles given by

$$z(K) = \sum_{\Delta \in K, \dim \Delta = \dim K - n} g(\text{Lk } \Delta, \dim K) \Delta$$

Such cycles are not characteristic local cycles because the function g depends on the dimension of K .

In section 2 we define the cochain complex $\mathcal{T}^*(\mathbb{Q})$ whose elements are skew-symmetric functions from the set of all isomorphism classes of oriented PL spheres to \mathbb{Q} . We prove that the function $f \in \mathcal{T}^*(\mathbb{Q})$ is a cocycle iff it is a local formula for a characteristic local cycle. We prove that there exists an isomorphism $H^*(\mathcal{T}^*(\mathbb{Q})) \cong H^*(BPL; \mathbb{Q}) = H^*(BO; \mathbb{Q})$. Hence any rational characteristic local cycle represents the homology classes dual to some polynomial in the Pontrjagin classes of the manifolds. Besides, a local formula for a given rational characteristic class is unique up to a coboundary.

In section 3 we obtain an explicit formula for all rational characteristic local cycles z such that the Poincaré dual of $z(K)$ represents the first Pontrjagin class of a combinatorial manifold K . This result is new because the formulae of [4, 5, 8] cannot be applied for an arbitrary combinatorial manifold and the formulae of [3] give only *real* characteristic local cycles. We use the following approach. First we find explicitly all rational characteristic local cycles of codimension 4, i. e. all local formulae $f \in \mathcal{T}^4(\mathbb{Q})$. Then we prove and use the theorem that any such characteristic local cycle represents the homology classes dual to the first Pontrjagin class multiplied by some rational constant. The usage of *bistellar moves* is very important.

In section 3.6 we find some estimates for the denominators of the local formulae's values. In particular, we prove that if f is a local formula for the first Pontrjagin class and q be an integer number, then there exists a PL sphere L such that the denominator of $f(L)$ is divisible by q . Hence there exist no integer local formulae representing nontrivial homology classes.

A brief formulation of our results is contained in [6]. In this paper we omit proofs of several propositions. Full proofs can be found in [7].

Terminology and notation. All necessary definitions and results of PL topology can be found in [15]. In the sequel all manifolds, triangulations and bordisms are supposed to be piecewise linear. All manifolds are supposed to be closed. All bordisms are supposed to be oriented. An isomorphism of oriented simplicial complexes is an isomorphism preserving orientation. An isomorphism

changing orientation is called an antiisomorphism. Let K be a simplicial complex on a set S . By CK , ΣK , and K' denote respectively the cone and the suspension over K , and the barycentric subdivision of K . The *full subcomplex* spanned by a subset $V \subset S$ is the subcomplex $L \subset K$ consisting of all simplices $\Delta \in K$ such that all vertices of Δ belong to V . By $\text{Lk } \Delta$ and $\text{Star } \Delta$ denote respectively the link and the star of a simplex Δ .

2 Local formulae

2.1 Main definitions and results

Let \mathcal{T}_n be the set of all isomorphism classes of oriented $(n-1)$ -dimensional PL spheres. Usually we don't distinguish between a PL sphere and its isomorphism class. For any $L \in \mathcal{T}_n$ by $-L$ denote the same triangulation with the opposite orientation. Let $L \in \mathcal{T}_n$ be *symmetric* if there exists an antiautomorphism of L . Let G be an abelian group. By $\mathcal{T}^n(G)$ denote the abelian group of all functions $f : \mathcal{T}_n \rightarrow G$ such that $f(L) = f(-L)$ for any $L \in \mathcal{T}_n$. We assume that $\mathcal{T}^0(G) = G$, $\mathcal{T}^{-n}(G) = 0$, $n > 0$. Let the differential $\delta : \mathcal{T}^n(G) \rightarrow \mathcal{T}^{n+1}(G)$ be given by $(\delta f)(L) = \sum f(\text{Lk } v)$, where the sum is over all vertices $v \in L$ and the orientation of $\text{Lk } v$ is induced by the orientation of L . Evidently, $\delta^2 = 0$. Thus $\mathcal{T}^*(G)$ is a cochain complex.

Let K be an m -dimensional combinatorial manifold. Let \widehat{G} be the local system on $|K|$ with fiber G and twisting given by the orientation. A coorientation of the simplex $\Delta^n \in K$ is an orientation of $\text{Lk } \Delta^n$. Any m -simplex is supposed to be positively cooriented. By $\widehat{C}_n(K; G)$ denote the complex of cooriented simplicial chains of K . Let $\widehat{\partial} : \widehat{C}_n(K; G) \rightarrow \widehat{C}_{n-1}(K; G)$ be the boundary operator. (The incidence coefficient of two simplices $\tau^{k-1} \subset \sigma^k$ is equal to $+1$ iff the orientation of $\text{Lk } \sigma^k$ is induced by the orientation of $\text{Lk } \tau^{k-1}$.) The homology of $\widehat{C}_*(K; G)$ is equal to $H_*(|K|; \widehat{G})$. If K is oriented then we have the augmentation $\epsilon : \widehat{C}_0(K; G) \rightarrow G$.

Consider $f \in \mathcal{T}^n(G)$. By $f_\#(K)$ denote the cooriented chain $f_\#(K) = \sum_{\Delta^{m-n} \in K} f(\text{Lk } \Delta^{m-n}) \Delta^{m-n} \in \widehat{C}_{m-n}(K; G)$. Evidently the addend $f(\text{Lk } \Delta^{m-n}) \Delta^{m-n}$ does not depend on the coorientation of Δ^{m-n} . Let $f \in \mathcal{T}^n(G)$ be a *local formula* if for any combinatorial manifold K the cooriented chain $f_\#(K)$ is a cycle.

Proposition 2.1. 1) f is a local formula if and only if f is a cocycle in the cochain complex $\mathcal{T}^*(G)$.

2) If f is a coboundary in $\mathcal{T}^*(G)$, then for any combinatorial manifold K the cycle $f_\#(K)$ is a boundary.

3) Let K_1 and K_2 be two triangulations of a manifold M^m . If f is a local formula, then $f_\#(K_1)$ and $f_\#(K_2)$ are homologous.

Proof. Notice that $\widehat{\partial} f_\#(K) = (\delta f)_\#(K)$ for any $f \in \mathcal{T}^n(G)$. The second claim of the proposition follows. Also, if f is a cocycle, then f is a local formula.

Suppose $\delta f \neq 0$. Then there exists $L \in \mathcal{T}_{n+1}$ such that $(\delta f)(L) \neq 0$. If $m > n$, then there exists an m -dimensional combinatorial manifold K such that $\text{Lk } \Delta \cong L$ for some simplex $\Delta \in K$. Then the coefficient at the simplex Δ in the chain $\widehat{\partial} f_\#(K) = (\delta f)_\#(K)$ is nonzero. Hence $f_\#(K)$ is not a cycle. Therefore f is not a local formula.

Let us now prove the third claim of the proposition. Assume that $m > n$. K_1 can be transformed to K_2 by the sequence of stellar subdivisions and inverse stellar subdivisions (see [1]). We can assume without loss of generality that K_1 can be transformed to K_2 by a stellar subdivision of some simplex Δ . Then the support of the cycle $f_{\sharp}(K_2) - f_{\sharp}(K_1)$ is contained in the subcomplex $\text{Star } \Delta$, which is contractible. The proposition follows.

The case $n = m$ is proved in section 2.2. \square

Consider $\psi \in H^n(\mathcal{T}^*(G))$. For any manifold M^m by $\psi_{\sharp}(M^m) \in H_{m-n}(M^m; \widehat{G})$ denote the homology class represented by $f_{\sharp}(K)$, where K is an arbitrary triangulation of M^m and f is an arbitrary representative of ψ . By $\psi^{\sharp}(M^m) \in H^n(M^m; G)$ denote the Poincaré dual of $\psi_{\sharp}(M^m)$. If M^m is oriented and $m = n$, then let $\psi^*(M^n)$ be given by $\psi^*(M^n) = \langle \psi^{\sharp}(M^n), [M^n] \rangle$.

Theorem 2.1. *For any rational characteristic class $p \in H^n(BPL; \mathbb{Q})$ there exists a unique cohomology class $\phi_p \in H^n(\mathcal{T}^*(\mathbb{Q}))$ such that $\phi_p^{\sharp}(M) = p(M)$ for any manifold M . The homomorphism $H^*(BPL; \mathbb{Q}) \rightarrow H^*(\mathcal{T}^*(\mathbb{Q}))$ given by $p \mapsto \phi_p$ is an isomorphism.*

Corollary 2.1. *There is an additive isomorphism $H^*(\mathcal{T}^*(\mathbb{Q})) \cong \mathbb{Q}[p_1, p_2, \dots]$, $\deg p_j = 4j$.*

Corollary 2.2. *If $f \in \mathcal{T}^*(\mathbb{Q})$ and $f_{\sharp}(K)$ is a boundary for any combinatorial manifold K , then f is a coboundary.*

2.2 Invariance under bordisms

Proposition 2.2. *Let $\psi \in H^n(\mathcal{T}^*(G))$ be an arbitrary cohomology class. Then $\psi^*(M_1^n) = \psi^*(M_2^n)$ for any two bordant oriented manifolds M_1^n and M_2^n .*

Let $f \in \mathcal{T}^n(G)$ be a local formula. Let L be an n -dimensional null cobordant oriented combinatorial manifold. Prove that $\epsilon(f_{\sharp}(L)) = 0$. Proposition 2.2 and the unproved case of Proposition 2.1 follow.

Let K be an $(n+1)$ -dimensional combinatorial manifold with boundary such that $\partial K = L$. Suppose u is a cooriented vertex of L ; then $\text{Lk}_K u$ is a triangulation of n -disk. The coorientation of the vertex u in the simplicial complex K induces the coorientation of the vertex u in the complex L . Hence we have the monomorphism $\widehat{C}_0(L; G) \rightarrow \widehat{C}_0(K; G)$. Obviously, $\partial \text{Lk}_K u = \text{Lk}_L u$. By $\text{Lk}_K^* u$ denote the simplicial complex $\text{Lk}_K u \cup_{\text{Lk}_L u} C(\text{Lk}_L u)$ whose orientation is induced by the orientation of $\text{Lk}_K u$. Then $\text{Lk}_K^* u \in \mathcal{T}_{n+1}$. Similarly, for any cooriented 1-simplex $e \in L$ define the PL sphere $\text{Lk}_K^* e \in \mathcal{T}_n$.

Let the 1-chain $f_{\sharp}(K) \in \widehat{C}_1(K; G)$ be given by $f_{\sharp}(K) = \sum f(\text{Lk}_K^* e)e + \sum f(\text{Lk}_K e)e$ where the first sum is over all 1-simplices $e \in L$ and the second sum is over all 1-simplices $e \in (K \setminus L)$. Since f is a local formula, we have $(\delta f)(\text{Lk}_K^* u) = 0$ for any vertex $u \in L$. Therefore $\sum f(\text{Lk}_K^* e) + \sum f(\text{Lk}_K e) = f(\text{Lk}_L u)$, where the first sum is over all 1-simplices $e \in L$ such that $u \in e$ and the second sum is over all 1-simplices $e \in K \setminus L$ such that $u \in e$ (the coorientation of e is chosen so that the incidence coefficient of the pair (v, e) is equal to $+1$). Consequently $\widehat{\partial} f_{\sharp}(K) = i(f_{\sharp}(L))$. Hence $\epsilon(f_{\sharp}(L)) = 0$.

Note 2.1. Indeed, the formula $\widehat{\partial} f_{\sharp}(K) = i(f_{\sharp}(\partial K))$ holds for a combinatorial manifold K of any dimension $m \geq n + 1$.

2.3 The isomorphism \star

Let Ω_n be a group of oriented n -dimensional PL bordisms of a point. From Proposition 2.2 it follows that the homomorphism $\star : H^n(\mathcal{T}^*(G)) \rightarrow \text{Hom}(\Omega_n, G)$ taking each cohomology class ψ to ψ^\star is well defined. There exists a canonical isomorphism $\text{Hom}(\Omega_n, \mathbb{Q}) \cong H^n(BPL; \mathbb{Q})$. Hence we have a homomorphism $\sharp : H^n(\mathcal{T}^*(\mathbb{Q})) \rightarrow H^n(BPL; \mathbb{Q})$.

Theorem 2.2. *The homomorphism $\star : H^n(\mathcal{T}^*(\mathbb{Q})) \rightarrow \text{Hom}(\Omega_n, \mathbb{Q})$ is an isomorphism.*

Corollary 2.3. *The homomorphism $\sharp : H^n(\mathcal{T}^*(\mathbb{Q})) \rightarrow H^n(BPL; \mathbb{Q})$ is an isomorphism.*

The fact that \star is an epimorphism can be easily deduced from the results of Levitt and Rourke [13]. Their approach uses an explicit construction of the classifying space $B\widehat{PL}_n$ for block bundles. However it is possible to prove that \star is an isomorphism quite elementary, see [7]. Here we omit the proof of the fact that \star is a monomorphism, see [7].

2.4 The cohomology classes $\psi^\sharp(M^m)$ for manifolds of an arbitrary dimension

From Proposition 2.2 it follows that for any cohomology class $\psi \in H^n(\mathcal{T}^*(\mathbb{Q}))$ there exists a rational characteristic class $p = \sharp(\psi)$ such that $\psi^\sharp(M^n) = p(M^n)$ for any n -dimensional manifold M^n . Let us prove that this formula holds for a manifold of any dimension.

Proposition 2.3. *$\psi^\sharp(M^m) = p(M^m) = \sharp(\psi)(M^m)$ for any manifold M^m , $m \geq n$.*

Obviously, Theorem 2.1 follows from Corollary 2.3 and Proposition 2.3.

Let P be a compact polyhedron. Let $Q \subset P$ be a closed PL subset. We say that P is a manifold with singularities in Q if $P \setminus Q$ is a (nonclosed) manifold. We have the Lefschetz duality $H_{m-n}(P, Q; \widehat{G}) \cong H^n(P \setminus Q; G)$ for $n < m$. Let $\psi \in H^n(\mathcal{T}^*(\mathbb{Q}))$ be a cohomology class satisfying $n < m$. Consider an arbitrary triangulation K of P . Let $L \subset K$ be the subcomplex consisting of all closed simplices whose intersections with Q are not empty. Let $f \in \mathcal{T}^n(G)$ be a local formula representing ψ . Let the chain $f_\sharp(K, L) \in \widehat{C}_{m-n}(K, L; G)$ be given by $\sum f(\text{Lk } \Delta^{m-n}) \Delta^{m-n}$, where the sum is over all $(m-n)$ -simplices $\Delta^{m-n} \in K \setminus L$. Arguing as in the proof of Proposition 2.1 it is easy to prove that $f_\sharp(K, L)$ is a relative cycle, whose homology class does not depend on the choice of a triangulation K and a local formula f representing ψ . Thus the classes $\psi_\sharp(P, Q) \in H_{m-n}(P, Q; \widehat{G})$ and $\psi^\sharp(P, Q) \in H^n(P \setminus Q; G)$ are well defined. Let $S \subset P \setminus Q$ be a compact subset. Evidently, the cohomology class $\psi^\sharp(P, Q)|_S$ is determined by the topology of some neighborhood $U \supset S$. Hence we have the following proposition.

Proposition 2.4. *Let N^k be a (closed) oriented manifold. Let $\psi \in H^n(\mathcal{T}^*(G))$, $n \leq k$. Consider an m -dimensional manifold P with singularities in Q . Let $i : N^k \hookrightarrow P \setminus Q$ be an embedding such that $i(N^k) \subset P \setminus Q$ is a submanifold with trivial normal bundle. Put $\psi_m^\sharp(N^k) = i^*(\psi^\sharp(K, L))$. Then the cohomology*

class $\psi_m^\sharp(N^k)$ depends only on the manifold N^k and the number m and does not depend on the choice of the triple (P, Q, i) .

Proposition 2.5. $\psi_m^\sharp(N^k) = \psi^\sharp(N^k)$ for any $m > k$.

Proof. The join $N^k * \Delta^{m-k-1}$ is an m -dimensional manifold with singularities in $N^k \sqcup \Delta^{m-k-1}$. Points of $N^k * \Delta^{m-k-1}$ are linear combinations $t_0x + \sum_{j=1}^{m-k} t_j y_j$, where y_1, y_2, \dots, y_{m-k} are the vertices of the simplex Δ^{m-k-1} , $x \in N^k$, $t_j \geq 0$, $j = 0, 1, \dots, m-k$, $\sum_{j=0}^{m-k} t_j = 1$. Let $i : N^k \hookrightarrow N^k * \Delta^{m-k-1}$ be the embedding given by $i(x) = \frac{1}{m-k+1}(x + \sum_{j=1}^{m-k} y_j)$. Then $i(N^k)$ is a submanifold with trivial normal bundle. Let K be an arbitrary triangulation of N^k . The embedding i is transversal to simplices of K . We have $|\tau * \Delta^{m-k-1}| \cap i(N^k) = i(|\tau|)$ and $\text{Lk}_{K * \Delta^{m-k-1}}(\tau * \Delta^{m-k-1}) = \text{Lk}_K \tau$ for any simplex $\tau \in K$. Hence for any local formula f the intersection of the cycles $f_\sharp(K * \Delta^{m-k-1}, K \sqcup \Delta^{m-k-1})$ and $i_*([N^k])$ coincides with the cycle $i_*(f_\sharp(K))$. Therefore $\psi_m^\sharp(N^k) = \psi^\sharp(M^m)|_{N^k} = \psi^\sharp(N^k)$. \square

Proof of Proposition 2.3. Consider the case $G = \mathbb{Q}$, $n = k$. Let M^m be an oriented manifold such that $m > n$. By Propositions 2.4 and 2.5 we have $\psi^\sharp(M^m)|_{N^n} = p(N^n)$ for any submanifold $N^n \subset M^m$ with trivial normal bundle. From the results of Rokhlin, Schwarz and Thom it follows that $\psi^\sharp(M^m) = p(M^m)$ if $m > 2n + 1$.

Assume that $n < m \leq 2n + 1$. Let $i : M^m \hookrightarrow M^m \times S^{n+1}$ be the standard embedding. By Propositions 2.4 and 2.5 we have $i^*(\psi^\sharp(M^m \times S^{n+1})) = \psi^\sharp(M^m)$. On the other hand, $i^*(p(M^m \times S^{n+1})) = p(M^m)$ and $\psi^\sharp(M^m \times S^{n+1}) = p(M^m \times S^{n+1})$. Therefore $\psi^\sharp(M^m) = p(M^m)$.

Assume now that M^m is not orientable. Let $\pi : \widetilde{M}^m \rightarrow M^m$ be the oriented two-fold covering. Then $\pi^*(\psi^\sharp(M^m)) = \psi^\sharp(\widetilde{M}^m)$, $\pi^*(p(M^m)) = p(\widetilde{M}^m)$. The proposition follows because π^* is a monomorphism. \square

3 Explicit local formulae for the first Pontrjagin classes

3.1 Graphs Γ_n and their cohomology

Let K be a combinatorial manifold. Assume that there is a simplex $\Delta_1 \in K$ such that $\text{Lk } \Delta_1 = \partial \Delta_2$. (Obviously, $\Delta_2 \notin K$.) Then $\Delta_1 * \partial \Delta_2$ is a full subcomplex of K . The *bistellar move* associating with the simplex Δ_1 is the operation taking K to the simplicial complex $(K \setminus (\Delta_1 * \partial \Delta_2)) \cup (\partial \Delta_1 * \Delta_2)$. If $\dim \Delta = 0$, then $\partial \Delta = \emptyset$. We assume that $\Delta * \emptyset = \Delta$ for any simplex Δ . Hence the bistellar move associating with a maximal simplex of K is just the stellar subdivision of this simplex. The bistellar move associating with a 0-simplex is the inverse stellar subdivision. By a theorem of Pachner [14] (see also [2]) two PL manifolds are PL homeomorphic if and only if there exists a sequence of bistellar moves transforming the first manifold to the second one.

Consider two PL spheres $L_1, L_2 \in \mathcal{T}_{n+1}$. Let β_1 and β_2 be two bistellar moves transforming L_1 to L_2 such that β_1 and β_2 are associated with simplices Δ_1 and Δ_2 respectively. We say that β_1 and β_2 are *equivalent* if there exists an automorphism of L_1 taking the simplex Δ_1 to Δ_2 . In the sequel we do

not distinguish between a bistellar move and its equivalence class. For any bistellar move β by β^{-1} denote the inverse bistellar move. If a bistellar move β transforming some PL sphere L to itself is equivalent to the bistellar move β^{-1} , then the bistellar move β is said to be *inessential*. Let a bistellar move be *essential* if it is not inessential.

Let us construct the graph Γ_n in the following way. The vertex set of Γ_n is the set \mathcal{T}_{n+1} . To each equivalence class β of essential bistellar moves transforming L_1 to L_2 assign the edge e_β of Γ_n with endpoints L_1 and L_2 . The edge $e_{\beta^{-1}}$ corresponding to the inverse bistellar move coincides with the edge e_β but has the opposite orientation. By Pachner's theorem the graph Γ_n is connected.

Let $C_*(\Gamma_n; \mathbb{Z})$ be the cellular chain complex of the graph Γ_n . We mean that $e_\beta = 0 \in C_1(\Gamma_n; \mathbb{Z})$ if the bistellar move β is inessential. The group \mathbb{Z}_2 acts on the graph Γ_n by changing orientation of all PL spheres. The group \mathbb{Z}_2 acts on the group \mathbb{Q} by changing sign. Let $C_{\mathbb{Z}_2}^*(\Gamma_n; \mathbb{Q}) = \text{Hom}_{\mathbb{Z}_2}(C_*(\Gamma_n; \mathbb{Z}), \mathbb{Q})$ be the equivariant cochain complex of Γ_n . Let $H_{\mathbb{Z}_2}^*(\Gamma_n; \mathbb{Q}) = H^*(C_{\mathbb{Z}_2}^*(\Gamma_n; \mathbb{Q}))$ be the equivariant cohomology of Γ_n . By d denote the differential of the complex $C_{\mathbb{Z}_2}^*(\Gamma_n; \mathbb{Q})$. We have the isomorphism $H_{\mathbb{Z}_2}^1(\Gamma_n; \mathbb{Q}) \cong \text{Hom}_{\mathbb{Z}_2}(H_1(\Gamma_n; \mathbb{Z}), \mathbb{Q})$.

Evidently, $C_{\mathbb{Z}_2}^0(\Gamma_{n-1}; \mathbb{Q}) = \mathcal{T}^n(\mathbb{Q})$. Hence we have the differential $\delta : C_{\mathbb{Z}_2}^0(\Gamma_{n-1}; \mathbb{Q}) \rightarrow C_{\mathbb{Z}_2}^0(\Gamma_n; \mathbb{Q})$, $\delta^2 = 0$. Consider a bistellar move β transforming L_1 to L_2 , $L_1, L_2 \in \mathcal{T}_{n+1}$. We may assume that L_1 and L_2 are simplicial complexes on the same vertex set V . It is actually true if β is associated with the simplex, whose dimension is neither 0 nor n . Otherwise we assume that one of the two simplicial complexes contains the vertex v_0 which is not a simplex of this complex. Let W be the set of all vertices $v \in V$ such that the bistellar move β induces an essential bistellar move β_v transforming $\text{Lk}_{K_1} v$ to $\text{Lk}_{K_2} v$ (by definition, $v_0 \notin W$). Let the differential $\delta : C_{\mathbb{Z}_2}^1(\Gamma_{n-1}; \mathbb{Q}) \rightarrow C_{\mathbb{Z}_2}^1(\Gamma_n; \mathbb{Q})$ be given by $(\delta h)(e_\beta) = \sum_{v \in W} h(e_{\beta_v})$. It is easy to prove that $\delta^2 = 0$ and $\delta d = d\delta$.

Put $C^{j,n} = C_{\mathbb{Z}_2}^j(\Gamma_{n-1}; \mathbb{Q})$. Then $C^{*,*}$ is a bigraded complex. We have $\text{bideg } d = (1, 0)$ and $\text{bideg } \delta = (0, 1)$. By $Z_d^{*,*}$, $B_d^{*,*}$, and $H_d^{*,*}$ denote respectively the cocycle group, the coboundary group, and the cohomology group of the complex $C^{*,*}$ with respect to the differential d . By $Z_\delta^{*,*}$, $B_\delta^{*,*}$, $H_\delta^{*,*}$ denote respectively the cocycle group, the coboundary group, and the cohomology group of the complex $C^{*,*}$ with respect to the differential δ . The graph Γ_{n-1} is connected. Hence $H_d^{0,n} = 0$. Therefore $d : C^{0,n} \rightarrow C^{1,n}$ is a monomorphism.

Consider $L_1, L_2 \in \mathcal{T}_n$. Let β , V , and W be as above. Consider the cone CL_1 with vertex u_1 and the cone CL_2 with vertex u_2 . Let L_β be the simplicial complex on the set $V \cup \{u_1, u_2\}$ given by $L_\beta = CL_1 \cup CL_2 \cup (\Delta_1 * \Delta_2)$. Choose the orientation of L_β such that the induced orientation of $\text{Lk } u_2$ coincides with the orientation of L_2 . Then $L_\beta \in \mathcal{T}_{n+1}$. If β_1 and β_2 are equivalent bistellar moves, then the complexes L_{β_1} and L_{β_2} are isomorphic. The complexes L_β and $L_{\beta^{-1}}$ are antiisomorphic. If β is inessential, then L_β is symmetric. Let the homomorphism $s : C^{0,n} \rightarrow C^{1,n-1}$ be given by $s(f)(e_\beta) = f(L_\beta)$.

Obviously, $d : C^{0,*} \rightarrow C^{1,*}$ is a chain homomorphism of the complexes with differential δ .

Proposition 3.1. *The homomorphism s is a chain homotopy between the chain homomorphisms d and 0 of $C^{0,*}$ to $C^{1,*}$, i. e. $d = \delta s + s\delta$.*

Proof. For any $v \in V \setminus W$ the link of the vertex v in L_β is symmetric. For any $v \in W$ the link of the vertex v in L_β is isomorphic to $-L_{\beta_v}$. The links of the

vertices u_1 and u_2 are isomorphic to $-L_1$ and L_2 respectively. Hence for any $f \in C^{0,n}$ we have

$$\begin{aligned} s(\delta f)(e_\beta) &= (\delta f)(L_\beta) = - \sum_{v \in W} f(L_{\beta_v}) + f(L_2) - f(L_1) \\ &= - \sum_{v \in W} s(f)(e_{\beta_v}) + f(\partial e_\beta) = -\delta s(f)(e_\beta) + df(e_\beta) \end{aligned}$$

Consequently $df = \delta s(f) + s(\delta f)$. \square

Let A^n be the subgroup of $C^{1,n}$ consisting of all h such that $\delta h \in B_d^{1,n+1}$.

Proposition 3.2. *The homomorphism $s|_{Z_\delta^{0,n}}$ is a monomorphism and $s(Z_\delta^{0,n}) \subset A^{n-1}$.*

Proof. By Proposition 3.1, $d|_{Z_\delta^{0,n}} = \delta s|_{Z_\delta^{0,n}}$. The homomorphism d is a monomorphism. Therefore $s|_{Z_\delta^{0,n}}$ is a monomorphism. If $f \in Z_\delta^{0,n}$, then $\delta s(f) = df$. Hence $s(f) \in A^{n-1}$. \square

Corollary 3.1. $Z_\delta^{0,3} = 0$. Hence $H_\delta^{0,3} = 0$.

Proof. The graph Γ_1 is isomorphic to the graph with the vertex set $\{3, 4, 5, 6, \dots\}$ such that for any k there exists a unique edge with the endpoints k and $(k+1)$. The action of the group \mathbb{Z}_2 is trivial. Therefore $C^{1,2} = 0$. By Proposition 3.2, there exists a monomorphism of $Z_\delta^{0,3}$ to $C^{1,2}$. Hence $Z_\delta^{0,3} = 0$. \square

Proposition 3.3. $s|_{B_\delta^{0,4}}$ is an isomorphism of $B_\delta^{0,4}$ to $B_d^{1,3}$.

Proof. We have $C^{1,2} = 0$. Therefore $dg = s(\delta g)$ for any $g \in C^{0,3}$. The proposition follows. \square

Thus s induces the monomorphism s^* of $H_\delta^{0,4}$ to $H_d^{1,3} = H_{\mathbb{Z}_2}^1(\Gamma_2; \mathbb{Q})$. Let \tilde{A}^3 be the kernel of the homomorphism $\delta^* : H^{1,3} \rightarrow H^{1,4}$ induced by the chain homomorphism $\delta : C^{*,3} \rightarrow C^{*,4}$. Then $s^*(H_\delta^{0,4}) \subset \tilde{A}^3$.

3.2 Generators of the group $H_1(\Gamma_2; \mathbb{Z})$

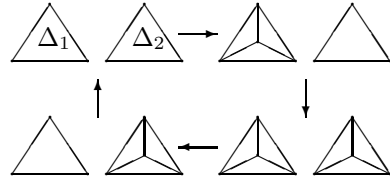
In the sequel if we say that $\{u_0, \dots, u_l\}$ is a simplex of an l -dimensional oriented simplicial complex L , then we mean that the sequence of the vertices u_0, \dots, u_l provides the given orientation of L . If we show a 2-dimensional simplicial complex in a figure, then we mean that the orientation is clockwise. For any cycle $\gamma \in Z_1(\Gamma_n; \mathbb{Z})$ let $\bar{\gamma} \in H_1(\Gamma_n; \mathbb{Z})$ be the homology class represented by γ .

Let L be an oriented 2-dimensional PL sphere. An edge $e \in L$ is called *admissible* if there exists a bistellar move associated with e . A pair of edges (e_1, e_2) is called *admissible* if:

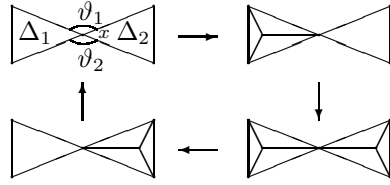
- 1) there is no triangle $\Delta \in K$ containing both e_1 and e_2 ;
- 2) e_1 and e_2 are admissible;
- 3) the bistellar move associated with the edge e_1 takes e_2 to an admissible edge.

Let $\Delta_1, \Delta_2 \in L$ be two distinct triangles. Apply to L the bistellar move associated with Δ_1 ; by v_1 denote the new vertex created by this bistellar move.

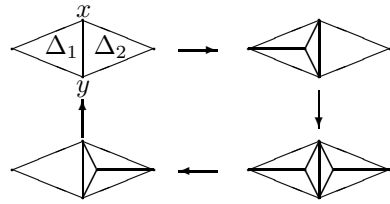
Then apply to the obtained triangulation the bistellar move associated with Δ_2 ; by v_2 denote the new vertex created by this bistellar move. Apply to the obtained triangulation the bistellar move associated with the vertex v_1 . Finally, apply to the obtained triangulation the bistellar move associated with the vertex v_2 . By $\alpha_1(L, \Delta_1, \Delta_2)$ denote the obtained cycle in the graph Γ_2 , see Fig. 1, a, b, c.



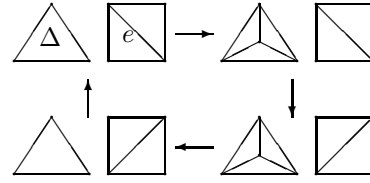
a



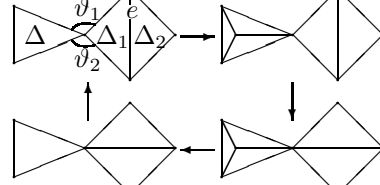
b



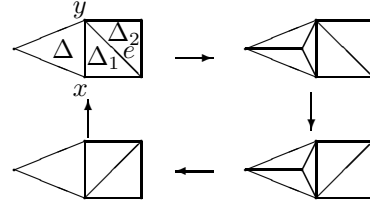
c



a



b



c

Fig. 1

Fig. 2

There are three possibilities: the triangles Δ_1 and Δ_2 can have no common vertices or have 1 or 2 common vertices. Let \mathcal{S}_1^0 be the set of all homology classes $\bar{\alpha}_1(L, \Delta_1, \Delta_2) \in H_1(\Gamma_2, \mathbb{Z})$ such that the triangles Δ_1 and Δ_2 have no common vertices, see Fig. 1, a.

By $\mathcal{S}_1^1(p, q)$ denote the set of all homology classes $\bar{\alpha}_1(L, \Delta_1, \Delta_2)$ such that:

- 1) the triangles Δ_1 and Δ_2 have a unique common vertex x , see Fig. 1, b;
- 2) there are exactly p triangles containing the vertex x and situated in the angle ϑ_1 ;
- 3) there are exactly q triangles containing the vertex x and situated in the angle ϑ_2 .

By $\mathcal{S}_1^2(p, q)$ denote the set of all homology classes $\bar{\alpha}_1(L, \Delta_1, \Delta_2)$ such that:

- 1) the triangles Δ_1 and Δ_2 have a common edge e with endpoints x and y , see Fig. 1, c;
- 2) there are exactly p triangles that contain the vertex x and coincide neither with Δ_1 nor with Δ_2 ;
- 3) there are exactly q triangles that contain the vertex y and coincide neither with Δ_1 nor with Δ_2 .

Let $\Delta \in L$ be a triangle. Let $e \in L$ be an admissible edge such that $e \not\subset \Delta$. By $\alpha_2(L, \Delta, e)$ denote the cycle in Fig. 2, a, b, c. Let \mathcal{S}_2^0 be the set of all homology classes $\bar{\alpha}_2(L, \Delta, e)$ such that the triangle Δ has common vertices

neither with Δ_1 nor with Δ_2 , where Δ_1 and Δ_2 are the two triangles containing the edge e , see Fig. 2, a.

By $\mathcal{S}_2^1(p, q)$ denote the set of all homology classes $\bar{\alpha}_2(L, \Delta, e)$ such that:

- 1) the triangle Δ has a unique common vertex x with the triangle Δ_1 , see Fig. 2, b;
- 2) there are exactly p triangles containing the vertex x and situated in the angle ϑ_1 ;
- 3) there are exactly q triangles containing the vertex x and situated in the angle ϑ_2 .

By $\mathcal{S}_2^2(p, q)$ denote the set of all homology classes $\bar{\alpha}_2(L, \Delta, e)$ such that:

- 1) the triangles Δ and Δ_1 have a common edge e_1 with endpoints x and y , see Fig. 2, c;
- 2) there are exactly p triangles that contain the vertex x and coincide neither with Δ nor with Δ_1 ;
- 3) there are exactly q triangles that contain the vertex y and coincide with none of the triangles Δ , Δ_1 , and Δ_2 .

Assume that (e_1, e_2) is an admissible pair. By $\alpha_3(L, e_1, e_2)$ denote the cycle shown in Fig. 3, a, b, c. Let \mathcal{S}_3^0 be the set of all homology classes $\bar{\alpha}_3(L, e_1, e_2)$ such that any triangle containing the edge e_1 has no common vertices with any triangle containing the edge e_2 , see Fig. 3, a.

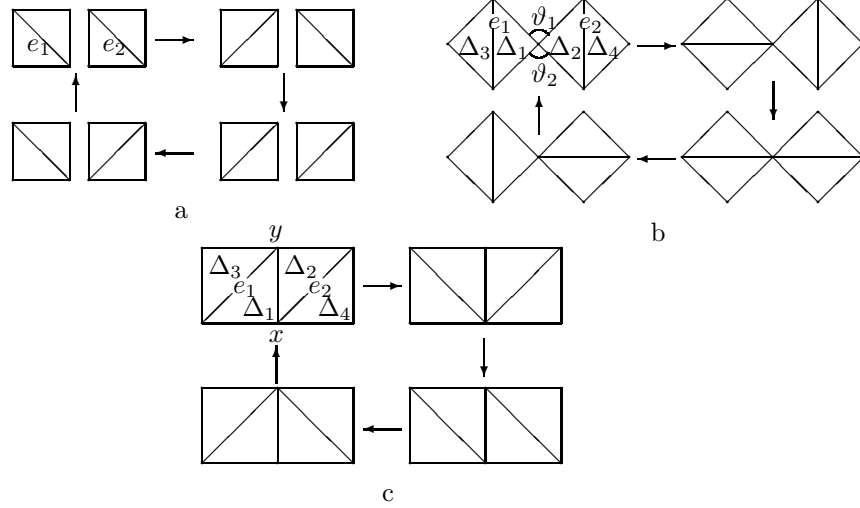


Fig. 3

By $\mathcal{S}_3^1(p, q)$ denote the set of all homology classes $\bar{\alpha}_3(L, e_1, e_2)$ such that:

- 1) the triangle Δ_1 has a unique common vertex x with the triangle Δ_2 , see Fig. 3, b;
- 2) there are exactly p triangles containing the vertex x and situated in the angle ϑ_1 ;
- 3) there are exactly q triangles containing the vertex x and situated in the angle ϑ_2 .

By $\mathcal{S}_3^2(p, q)$ denote the set of all homology classes $\bar{\alpha}_3(L, e_1, e_2)$ such that:

- 1) the triangles Δ_1 and Δ_2 have a common edge with endpoints x and y , see Fig. 3, c;
- 2) there are exactly p triangles that contain the vertex x and coincide with none of the triangles Δ_1 , Δ_2 , and Δ_4 ;

3) there are exactly q triangles that contain the vertex y and coincide with none of the triangles Δ_1 , Δ_2 , and Δ_3 .

Let x, y, z be vertices of L . Suppose there exists the vertex u such that $\{u, x, y\}$, $\{u, y, z\}$, and $\{u, z, x\}$ are triangles of L . Then by $\alpha_4(L, x, y, z)$ denote the cycle shown in Fig. 4. Let $\mathcal{S}_4(p, q, r)$ be the set of all homology classes $\bar{\alpha}_4(L, x, y, z)$ such that there are exactly p, q , and r triangles that contain respectively the vertices x, y , and z and coincide with none of the triangles $\{u, x, y\}$, $\{u, y, z\}$, and $\{u, z, x\}$.

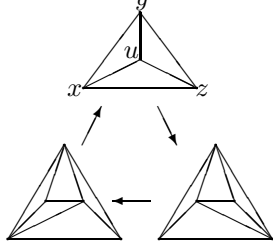


Fig. 4

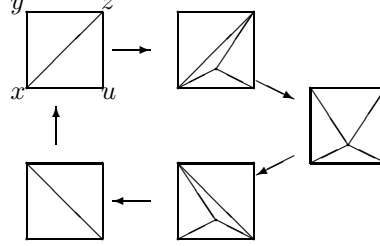


Fig. 5

Let x, y, z, u be the vertices of L . Suppose the full subcomplex of L spanned by the set $\{x, y, z, u\}$ consists of the triangles $\{x, y, z\}$ and $\{x, z, u\}$, their edges and vertices. Then let $\alpha_5(L, x, y, z, u)$ be the cycle shown in Fig. 5. Let $\mathcal{S}_5(p, q, r, k)$ be the set of all homology classes $\bar{\alpha}_5(L, x, y, z, u)$ such that there are exactly p, q, r , and k triangles that contain respectively the vertices x, y, z , and u and coincide neither with $\{x, y, z\}$ nor with $\{x, z, u\}$.

Let x, y, z, u, v be the vertices of L . Suppose the full subcomplex of L spanned by the set $\{x, y, z, u, v\}$ consists of the triangles $\{x, y, z\}$, $\{x, z, u\}$, and $\{x, u, v\}$, their edges and vertices. Then let $\alpha_6(L, x, y, z, u, v)$ be the cycle shown in Fig. 6. Let $\mathcal{S}_6(p, q, r, k, l)$ be the set of all homology classes $\bar{\alpha}_6(L, x, y, z, u, v)$ such that there are exactly p, q, r, k , and l triangles that contain respectively the vertices x, y, z, u , and v and coincide with none of the triangles $\{x, y, z\}$, $\{x, z, u\}$, and $\{x, u, v\}$.

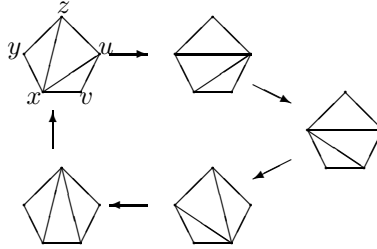


Fig. 6

By \mathcal{S} denote the union of all the sets $\mathcal{S}_1^0, \mathcal{S}_1^1(p, q), \mathcal{S}_1^2(p, q), \mathcal{S}_2^0, \mathcal{S}_2^1(p, q), \mathcal{S}_2^2(p, q), \mathcal{S}_3^0, \mathcal{S}_3^1(p, q), \mathcal{S}_3^2(p, q), \mathcal{S}_4(p, q, r), \mathcal{S}_5(p, q, r, k),$ and $\mathcal{S}_6(p, q, r, k, l)$.

Proposition 3.4. *The set \mathcal{S} generates the group $H_1(\Gamma_2; \mathbb{Z})$.*

3.3 The formula

By Theorem 2.1, there exists a unique generator $\phi \in H^4(T^*(\mathbb{Q})) \cong \mathbb{Q}$ such that $\phi^\sharp(M) = p_1(M)$ for any manifold M . The following theorem gives an explicit description of the generator ϕ .

Theorem 3.1. *Let $c_0 : \mathcal{S} \rightarrow \mathbb{Q}$ be the function given by*

$$c_0(\bar{\alpha}) = 0, \quad \bar{\alpha} \in \mathcal{S}_1^0 \cup \mathcal{S}_2^0 \cup \mathcal{S}_3^0$$

$$c_0(\bar{\alpha}) = \frac{q-p}{(p+q+2)(p+q+3)(p+q+4)}, \quad \bar{\alpha} \in \mathcal{S}_1^1(p, q) \cup \mathcal{S}_2^1(p, q) \cup \mathcal{S}_3^1(p, q)$$

$$c_0(\bar{\alpha}) = \frac{q}{(q+2)(q+3)(q+4)} - \frac{p}{(p+2)(p+3)(p+4)}, \quad \bar{\alpha} \in \mathcal{S}_1^2(p, q) \cup \mathcal{S}_3^2(p, q)$$

$$c_0(\bar{\alpha}) = \frac{q}{(q+2)(q+3)(q+4)} + \frac{p}{(p+2)(p+3)(p+4)}, \quad \bar{\alpha} \in \mathcal{S}_2^2(p, q)$$

$$c_0(\bar{\alpha}) = \frac{1}{(p+2)(p+3)} - \frac{1}{(q+2)(q+3)} + \frac{1}{(r+2)(r+3)} - \frac{1}{12}, \quad \bar{\alpha} \in \mathcal{S}_4(p, q, r)$$

$$c_0(\bar{\alpha}) = \frac{1}{(p+2)(p+3)} - \frac{1}{(q+2)(q+3)} - \frac{1}{(r+2)(r+3)} + \frac{1}{(k+2)(k+3)}, \quad \bar{\alpha} \in \mathcal{S}_5(p, q, r, k)$$

$$c_0(\bar{\alpha}) = \frac{1}{(p+2)(p+3)} + \frac{1}{(q+2)(q+3)} + \frac{1}{(r+2)(r+3)} + \frac{1}{(k+2)(k+3)} + \frac{1}{(l+2)(l+3)} - \frac{1}{12}, \quad \bar{\alpha} \in \mathcal{S}_6(p, q, r, k, l)$$

There exists a unique linear extension of c_0 to $H_1(\Gamma_2; \mathbb{Z})$. This extension, which we also denote by c_0 , belongs to the group $H_{\mathbb{Z}_2}^1(\Gamma_2; \mathbb{Q}) = \text{Hom}_{\mathbb{Z}_2}(H_1(\Gamma_2; \mathbb{Z}), \mathbb{Q})$. Then $c_0 = s^(\phi)$. Thus s maps isomorphically the affine space of all local formulae for the first Pontrjagin class to the affine space of all cocycles $\hat{c}_0 \in C_{\mathbb{Z}_2}^1(\Gamma_2; \mathbb{Q})$ representing the class c_0 .*

Note 3.1. Notice that similar numbers appear in [10] in a quite different problem. In this paper M. È Kazarian obtained a formula for a Chern-Euler class of a circle bundle over a closed surface in terms of singularities of a generic function's restrictions to the fibers.

How to make calculations using this theorem? First we need to choose a representative \hat{c}_0 of the class c_0 to fix a local formula f for the first Pontrjagin class. Let us now show how to calculate the value $f(L)$, where L is an oriented 3-dimensional PL sphere. Let $\beta_1, \beta_2, \dots, \beta_l$ be the sequence of bistellar moves transforming the boundary of 4-simplex to the PL sphere L . Let L_j be the PL sphere obtained from $\partial\Delta^4$ by applying the bistellar moves $\beta_1, \beta_2, \dots, \beta_{j-1}$. Let

W_j be the set of all vertices $v \in L_j$ such that the bistellar move β_j induces an essential bistellar move β_{jv} of $\text{Lk } v$. Then

$$f(L) = \sum_{j=1}^l \sum_{v \in W_j} (\widehat{c}_0)(e_{\beta_{jv}})$$

Assume that we need to calculate the first Pontrjagin number of an oriented 4-dimensional combinatorial manifold K . Then we don't need to choose a representative \widehat{c}_0 of the cohomology class c_0 . Let L^1, L^2, \dots, L^k be the links of all vertices of K . Define the bistellar moves β_j^i , the PL spheres L_j^i , the sets W_j^i and the bistellar moves β_{jv}^i as above.

Corollary 3.2. *The first Pontrjagin number of the manifold K is equal to*

$$\frac{1}{2} c_0 \left(\sum_{i=1}^k \sum_{j=1}^{l_i} \sum_{v \in W_j^i} (e_{\beta_{jv}^i} - \widetilde{e}_{\beta_{jv}^i}) \right)$$

where \widetilde{e} is the edge of the graph Γ_2 such that the action of the group \mathbb{Z}_2 takes e to \widetilde{e} .

Sketch of the proof of Theorem 3.1. Since $\star : H^4(\mathcal{T}^*(\mathbb{Q})) \rightarrow \text{Hom}(\Omega_4, \mathbb{Q})$ is an epimorphism, we see that $\dim H^4(\mathcal{T}^*(\mathbb{Q})) \geq 1$. On the other hand, in section 3.4 we prove that for any cohomology class $c \in \widetilde{A}^3$ there exists $\lambda \in \mathbb{Q}$ such that $c(\bar{\alpha}) = \lambda c_0(\bar{\alpha})$ for any $\bar{\alpha} \in \mathcal{S}$. Therefore $\dim \widetilde{A}^3 \leq 1$. But $s^* : H^4(\mathcal{T}^*(\mathbb{Q})) \rightarrow \widetilde{A}^3$ is a monomorphism. Hence $\dim H^4(\mathcal{T}^*(\mathbb{Q})) = \dim \widetilde{A}^3 = 1$. Therefore the cohomology class c_0 is well-defined, $c_0 \in \widetilde{A}^3$ and $s^*(\phi) = \lambda c_0$ for some rational $\lambda \neq 0$. In section 3.5 we prove that $\lambda = 1$. \square

3.4 The group \widetilde{A}^3

Consider an arbitrary $c \in \widetilde{A}^3 \subset \text{Hom}_{\mathbb{Z}_2}(H_1(\Gamma_2; \mathbb{Z}), \mathbb{Q})$.

Let $\alpha = \alpha_i(L, \dots)$ be a cycle shown in one of Fig. 1–6. By $X(\alpha)$ denote the set of all vertices that are denoted in the corresponding figure by one of the letters x, y, z, u , and v . The generator $\bar{\alpha} \in \mathcal{S}$ is called *regular* if it satisfies the following conditions:

- 1) $\bigcup_{a \in X(\alpha)} \text{Star } a$ is a full subcomplex of L .
- 2) If $w \notin X(\alpha)$, $a, b \in X(\alpha)$, and $\{w, a\}, \{w, b\} \in L$, then $\{w, a, b\} \in L$.

Consider $\bar{\alpha}_1(L, \Delta_1, \Delta_2) \in \mathcal{S}_1^0$. Let $K \in \mathcal{T}_4$ be a PL sphere containing a vertex u such that $\text{Lk } u \cong L$ and $\text{Star } u$ is a full subcomplex of K . Identify the two simplicial complexes $\text{Lk } u$ and L . Put $\widetilde{\Delta}_1 = \Delta_1 \cup \{u\}$ and $\widetilde{\Delta}_2 = \Delta_2 \cup \{u\}$. Apply to K the bistellar move associated with $\widetilde{\Delta}_1$; by z_1 denote the new vertex created by this bistellar move. Then apply to the obtained PL sphere the bistellar move associated with $\widetilde{\Delta}_2$; by z_2 denote the new vertex created by this bistellar move. Apply to the obtained PL sphere the bistellar move associated with the vertex z_1 . Finally, apply to the obtained PL sphere the bistellar move associated with the vertex z_2 . By γ denote the obtained cycle in the graph Γ_3 . For any vertex v of K the sequence of bistellar moves transforming K to itself induces the sequence of bistellar moves transforming $\text{Lk } v$ to itself. Hence

the cycle γ induces the cycle β_v in the graph Γ_2 . Then $\delta^*(c)(\bar{\gamma}) = \sum_v c(\bar{\gamma}_v)$. Now $\delta^*(c) = 0$ because $c \in \tilde{A}^3$. For any vertex $v \in K$ distinct from the vertex u the cycle γ_v is homologous to zero. The cycle γ_u coincides with the cycle $\alpha_1(L, \Delta_1, \Delta_2)$. Hence, $c(\bar{\alpha}_1(L, \Delta_1, \Delta_2)) = 0$.

Similarly, $c(\bar{\alpha}) = 0$ for any $\bar{\alpha} \in \mathcal{S}_2^0 \cup \mathcal{S}_3^0$.

Proposition 3.5. *For any $p, q > 0$ the function c is a constant function on the set $\mathcal{S}_1^1(p, q)$.*

Proof. Consider $\bar{\alpha}_1(L^{(i)}, \Delta_1^{(i)}, \Delta_2^{(i)}) \in \mathcal{S}_1^1(p, q)$, $i = 1, 2$. Let $x^{(i)}$ be the common vertex of the triangles $\Delta_1^{(i)}$ and $\Delta_2^{(i)}$. Assume that $\bar{\alpha}_1(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)})$ is regular. Obviously, there exists a 3-dimensional oriented PL sphere K containing an edge e with endpoints u and w such that the following conditions hold:

- 1) The link of e is a $(p + q + 2)$ -gon containing two edges denoted by e_1 and e_2 .
- 2) The link of u is isomorphic to $L^{(1)}$. This isomorphism takes the triangle spanned by the vertex w and the edge e_j to the triangle $\Delta_j^{(1)}$, $j = 1, 2$.
- 3) The link of w is isomorphic to $-L^{(2)}$. This isomorphism takes the triangle spanned by the vertex u and the edge e_j to the triangle $\Delta_j^{(2)}$, $j = 1, 2$.

By $\tilde{\Delta}_j$ denote the 3-simplex of L spanned by the edges e and e_j . Let the cycle γ be as above. For any vertex $v \in L$ distinct from the vertices u and w the induced cycle γ_v is homologous to zero. The cycle γ_u coincides with the cycle $\alpha_1(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(2)})$, the cycle γ_w coincides with the cycle $\alpha_1(-L^{(2)}, \Delta_1^{(2)}, \Delta_2^{(2)})$. Hence $c(\bar{\alpha}_1(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(2)})) = c(\bar{\alpha}_1(L^{(2)}, \Delta_1^{(2)}, \Delta_2^{(2)}))$. To conclude the proof note that each set $\mathcal{S}_1^1(p, q)$ contains a regular generator. \square

By $\rho(p, q)$ denote the value of the function c on the set $\mathcal{S}_1^1(p, q)$.

Proposition 3.6. *For any $p, q > 0$ the function c is a constant function on the set $\mathcal{S}_1^2(p, q)$. Let $\tau(p, q)$ be the value of the function c on the set $\mathcal{S}_1^2(p, q)$. Then $\tau(p, q) + \tau(q, r) + \tau(r, p) = 0$ for any $p, q, r > 0$.*

Proof. Consider $\bar{\alpha}_i(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)}) \in \mathcal{S}_1^2(p, q)$. Consider $r > 0$. Choose regular generators $\bar{\alpha}_1(L^{(2)}, \Delta_1^{(2)}, \Delta_2^{(2)}) \in \mathcal{S}_1^2(q, r)$ and $\bar{\alpha}_1(L^{(3)}, \Delta_1^{(3)}, \Delta_2^{(3)}) \in \mathcal{S}_1^2(r, p)$. There exists a 3-dimensional oriented PL sphere K containing a 2-simplex Δ_0 with vertices $u^{(1)}$, $u^{(2)}$, and $u^{(3)}$ such that the following conditions hold:

- 1) The link of $u^{(i)}$ is isomorphic to $L^{(i)}$.
- 2) This isomorphism takes the 2-dimensional face of the simplex $\tilde{\Delta}_j$ opposite the vertex $u^{(i)}$ to the triangle $\Delta_j^{(i)}$, $j = 1, 2$, where $\tilde{\Delta}_1, \tilde{\Delta}_2 \in L$ are the two tetrahedrons containing Δ_0 .

As in the prove of Proposition 3.5 we obtain

$$c(\bar{\alpha}_1(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)})) + c(\bar{\alpha}_1(L^{(2)}, \Delta_1^{(2)}, \Delta_2^{(2)})) + c(\bar{\alpha}_1(L^{(3)}, \Delta_1^{(3)}, \Delta_2^{(3)})) = 0$$

Instead of $\bar{\alpha}_1(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)})$ we can take any generator $\bar{\alpha} \in \mathcal{S}_1^2(p, q)$. Hence the function c is constant on $\mathcal{S}_1^2(p, q)$. The equality $\tau(p, q) + \tau(q, r) + \tau(r, p) = 0$ follows. \square

Evidently, $\tau(p, q) = -\tau(q, p)$. Therefore there exists a function $\chi : \mathbb{Z}_{>0} \rightarrow \mathbb{Q}$ such that $\tau(p, q) = \chi(q) - \chi(p)$ for any $p, q > 0$. The function χ is unique up to a constant. Extend the function ρ to the function of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ to \mathbb{Q} by putting $\rho(0, p) = \chi(p)$, $\rho(p, 0) = -\chi(p)$, and $\rho(0, 0) = 0$.

Proposition 3.7. *The function ρ satisfies the following equations.*

- (i) $\rho(p, q) = -\rho(q, p)$
- (ii) $\rho(p, q + r + 2) + \rho(q, r + p + 2) + \rho(r, p + q + 2) = \rho(p, q + r + 1) + \rho(q, r + p + 1) + \rho(r, p + q + 1)$

Proof. Equation (i) follows from $\alpha_1(L, \Delta_1, \Delta_2) = -\alpha_1(L, \Delta_2, \Delta_1)$.

Let L be an oriented simplicial 2-sphere containing a vertex x such that there exist exactly $(p + q + r + 3)$ triangles of L containing x . Let us go round the vertex x clockwise. Let Δ_1, Δ_2 , and Δ_3 be triangles containing the vertex x such that we pass through the triangle Δ_1 , then through r other triangles, then through the triangle Δ_2 , then through p other triangles, then through the triangle Δ_3 , then through q other triangles, and then again through the triangle Δ_1 . By L_j denote the PL sphere obtained from L by applying the bistellar move associated with Δ_j . It is easy to prove that $\sum_{j=1}^3 \alpha_1(L_j, \Delta_{j+1}, \Delta_{j+2}) = \sum_{j=1}^3 \alpha_1(L, \Delta_{j+1}, \Delta_{j+2})$, where the sums of indices are modulo 3. Applying c to the homology classes of both parts of this equality, we obtain equation (ii). \square

Proposition 3.8. *Suppose the function $\rho : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Q}$ satisfies equations (i) and (ii). Then there exist $b_1 \in \mathbb{Q}$ and $\lambda \in \mathbb{Q}$ such that $\rho(p, q) = \frac{\lambda(q-p)}{(p+q+2)(p+q+3)(p+q+4)}$ for any $p, q > 0$ and $\rho(0, q) = \frac{\lambda q}{(q+2)(q+3)(q+4)} + b_1$ for any $q > 0$.*

Without loss of generality we may assume that $\rho(p, q) = \frac{\lambda(q-p)}{(p+q+2)(p+q+3)(p+q+4)}$ for any $p, q \geq 0$.

The proofs of the following propositions are similar two the proofs of Propositions 3.5 and 3.6.

Proposition 3.9. $c(\bar{\alpha}) = \rho(p, q)$ for any $\bar{\alpha} \in \mathcal{S}_2^1(p, q) \cup \mathcal{S}_3^1(p, q)$.

Proposition 3.10. *There exists a constant $b_2 \in \mathbb{Q}$ such that $c(\bar{\alpha}) = \rho(0, q) + \rho(0, p) + b_2$ for any $\bar{\alpha} \in \mathcal{S}_2^2(p, q)$.*

Proposition 3.11. $c(\bar{\alpha}) = \rho(0, q) - \rho(0, p)$ for any $\bar{\alpha} \in \mathcal{S}_3^2(p, q)$.

Let $p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{25}, p_{34}, p_{35}, p_{36}, p_{45}, p_{46}, p_{56}$ be integers greater than 2. Let $\omega_j \in Z_1(\Gamma_2; \mathbb{Z})$, $j = 1, 2, \dots, 6$ be cycles such that the following conditions hold:

- 1) $\omega_1 = \alpha_4(L_1, x_1, y_1, z_1)$, $\bar{\omega}_1 \in \mathcal{S}_4(p_{13}, p_{14}, p_{12})$.
- 2) $\omega_2 = \alpha_5(L_2, x_2, y_2, z_2, u_2)$, $\bar{\omega}_2 \in \mathcal{S}_5(p_{23}, p_{12}, p_{24}, p_{25})$.
- 3) $\omega_3 = \alpha_6(L_3, x_3, y_3, z_3, u_3, v_3)$, $\bar{\omega}_3 \in \mathcal{S}_6(p_{34}, p_{13}, p_{23}, p_{35}, p_{36})$.
- 4) $\omega_4 = \alpha_6(L_4, x_4, y_4, z_4, u_4, v_4)$, $\bar{\omega}_4 \in \mathcal{S}_6(p_{34}, p_{46}, p_{45}, p_{24}, p_{14})$.
- 5) $\omega_5 = \alpha_5(L_5, x_5, y_5, z_5, u_5)$, $\bar{\omega}_5 \in \mathcal{S}_5(p_{45}, p_{56}, p_{35}, p_{25})$.
- 6) $\omega_6 = \alpha_4(L_6, x_6, y_6, z_6)$, $\bar{\omega} \in \mathcal{S}_4(p_{46}, p_{36}, p_{56})$;
- 7) At least 5 of the generators $\bar{\omega}_j$ are regular.

By $L_j^{(1)}$, $j = 1, 6$ denote the PL sphere obtained from L_j by the bistellar move associated with the vertex u_j .

There exists a 3-dimensional oriented PL sphere K containing vertices w_j , $j = 1, 2, \dots, 6$ such that the following conditions hold:

- 1) The full subcomplex of K spanned by the set $\{w_j, j = 1, \dots, 6\}$ consists of the tetrahedrons $\{w_1, w_2, w_3, w_4\}$, $\{w_2, w_3, w_4, w_5\}$, and $\{w_3, w_4, w_5, w_6\}$ and all

their faces.

2) For any edge $\{w_i, w_j\} \in L$, $i < j$ there are exactly p_{ij} tetrahedrons that contain the edge $\{w_i, w_j\}$ and coincide with none of the tetrahedrons $\{w_1, w_2, w_3, w_4\}$, $\{w_2, w_3, w_4, w_5\}$, and $\{w_3, w_4, w_5, w_6\}$.

3) The links of the vertices w_j can be identified with the complexes L_j for $1 < j < 6$ and $L_j^{(1)}$ for $j = 1, 6$ so that the vertices will be identified in the following way: $w_1 = y_2 = y_3 = v_4$, $w_2 = z_1 = z_3 = u_4 = u_5$, $w_3 = x_1 = x_2 = x_4 = z_5 = y_6$, $w_4 = y_1 = z_2 = x_3 = x_5 = x_6$, $w_5 = u_2 = u_3 = z_4 = z_6$, $w_6 = v_3 = y_4 = y_5$.

Apply to L the following sequence of bistellar moves:

- 1) Replace the 3-simplices $\{w_1, w_2, w_3, w_4\}$ and $\{w_2, w_3, w_4, w_5\}$ with the 3-simplices $\{w_1, w_2, w_3, w_5\}$, $\{w_1, w_3, w_4, w_5\}$, and $\{w_1, w_4, w_2, w_5\}$.
- 2) Replace the 3-simplices $\{w_1, w_3, w_4, w_5\}$ and $\{w_3, w_4, w_5, w_6\}$ with the 3-simplices $\{w_1, w_3, w_4, w_6\}$, $\{w_1, w_4, w_5, w_6\}$, and $\{w_1, w_5, w_3, w_6\}$.
- 3) Replace the 3-simplices $\{w_1, w_2, w_3, w_5\}$ and $\{w_1, w_5, w_3, w_6\}$ with the 3-simplices $\{w_2, w_3, w_1, w_6\}$, $\{w_2, w_1, w_5, w_6\}$, and $\{w_2, w_5, w_3, w_6\}$.
- 4) Replace the 3-simplices $\{w_1, w_4, w_2, w_5\}$, $\{w_1, w_6, w_4, w_5\}$, and $\{w_1, w_2, w_6, w_5\}$ with the 3-simplices $\{w_1, w_2, w_6, w_4\}$ and $\{w_2, w_6, w_4, w_5\}$.
- 5) Replace the 3-simplices $\{w_1, w_2, w_3, w_6\}$, $\{w_1, w_3, w_4, w_6\}$, and $\{w_1, w_4, w_2, w_6\}$ with the 3-simplices $\{w_1, w_2, w_3, w_4\}$ and $\{w_2, w_3, w_4, w_6\}$.
- 6) Replace the 3-simplices $\{w_2, w_3, w_4, w_6\}$, $\{w_2, w_4, w_5, w_6\}$, and $\{w_2, w_5, w_3, w_6\}$ with the 3-simplices $\{w_2, w_3, w_4, w_5\}$ and $\{w_3, w_4, w_5, w_6\}$.

This sequence of bistellar moves transforms L to itself. Let γ be the obtained cycle in the graph Γ_3 . The cycle γ_{w_j} is homologous to the cycle ω_j for $j = 3, 5, 6$. The cycle γ_{w_j} is homologous to the cycle $-\omega_j$ for $j = 1, 2, 4$. Therefore,

$$c(\bar{\omega}_1) + c(\bar{\omega}_2) - c(\bar{\omega}_3) + c(\bar{\omega}_4) - c(\bar{\omega}_5) - c(\bar{\omega}_6) = 0$$

Consequently the function c is constant on each of the sets $\mathcal{S}_4(p, q, r)$, $\mathcal{S}_5(p, q, r, k)$, and $\mathcal{S}_6(p, q, r, k, l)$, $p, q, r, k, l \geq 3$. By $\eta(p, q, r)$, $\zeta(p, q, r, k)$, and $\theta(p, q, r, k, l)$ respectively denote the values of the function c on these sets.

Proposition 3.12. *There exists a constant $b_5 \in \mathbb{Q}$ such that for any $p, q, r, k, l \geq 3$ we have*

$$\begin{aligned} \theta(p, q, r, k, l) = & \frac{\lambda}{(p+2)(p+3)} + \frac{\lambda}{(q+2)(q+3)} + \frac{\lambda}{(r+2)(r+3)} \\ & + \frac{\lambda}{(k+2)(k+3)} + \frac{\lambda}{(l+2)(l+3)} + b_5 \end{aligned}$$

Proof. Consider $\bar{\alpha}_6(L, x, y, z, u, v) \in \mathcal{S}_6(p, q, r, k, l)$, $p, q, r, k, l \geq 3$. Let $\Delta \in L$ be a triangle such that x is a vertex of Δ , y, z, u , and v are not vertices of Δ . Let us go round the vertex x clockwise. Suppose we pass through the triangle Δ , then through p' other triangles, then through the triangles $\{x, y, z\}$, $\{x, z, u\}$, $\{x, u, v\}$, then through p'' other triangles, and then again through the triangle Δ . Then $p' + p'' = p - 1$. The cycle $\alpha_6(L, x, y, z, u, v)$ is the sequence of 5 bistellar moves. By L_j , $j = 0, 1, 2, 3, 4$ denote the PL sphere obtained from L by the first j bistellar moves of this sequence. In particular, $L_0 = L$. By $L_j^{(1)}$ denote the simplicial complex obtained from L by the bistellar move associated with the triangle Δ . Let us define the graph G in the following way. The vertex set of G is the set $\{L_0, \dots, L_4, L_0^{(1)}, \dots, L_4^{(1)}\}$. To

the j -th bistellar move of the cycle $\alpha_6(L, x, y, z, u, v)$ assign the edge with endpoints L_{j-1} and L_j (or L_4 and L_0 if $j = 5$). Similarly, to the j -th bistellar move of the cycle $\alpha_6(L_0^{(1)}, x, y, z, u, v)$ assign the edge with endpoints $L_{j-1}^{(1)}$ and $L_j^{(1)}$ (or $L_4^{(1)}$ and $L_0^{(1)}$ if $j = 5$). For any $j = 0, 1, 2, 3, 4$ consider the bistellar move associated with the triangle Δ . To this bistellar move assign the edge with the endpoints L_j and $L_j^{(1)}$. There is the canonical map of the graph G to the graph Γ_2 . (This map is not necessarily an injection.) The graph G is isomorphic to a 1-skeleton of a pentagonal prism. Let us go the round of some 2-dimensional face of this sphere. We obtain the cycle in the graph Γ_2 . In this way, we obtain the cycles $\alpha_6(L, x, y, z, u, v)$, $-\alpha_6(L_0^{(1)}, x, y, z, u, v)$, $\alpha_2(L_0, \Delta, \{x, z\})$, $\alpha_2(L_1, \Delta, \{u, x\})$, $\alpha_2(L_2, \Delta, \{y, u\})$, $\alpha_2(L_3, \Delta, \{v, y\})$, $\alpha_2(L_4, \Delta, \{z, v\})$. The sum of all these cycles is equal to zero. Consequently,

$$\theta(p, q, r, k, l) - \theta(p+1, q, r, k, l) - \rho(p', p''+1) + \rho(p'+1, p'') = 0$$

Hence,

$$\theta(p+1, q, r, k, l) - \theta(p, q, r, k, l) = -\frac{2\lambda}{(p+2)(p+3)(p+4)}$$

In addition, the function θ is cyclically symmetric. The proposition follows from the equality

$$\sum_{i=1}^j \frac{1}{i(i+1)(i+2)} = \frac{1}{4} - \frac{1}{2(j+1)(j+2)}$$

□

Proposition 3.13. $b_2 = 0$.

Proof. Consider $\bar{\alpha}_6(L, x, y, z, u, v) \in \mathcal{S}_6(p, q, r, k, l)$, $p, q, r, k, l \geq 3$. Let Δ be the triangle that contains the edge $\{x, y\}$ and does not coincide with the triangle $\{x, y, z\}$. Arguing as in the proof of Proposition 3.12, we see that

$$\begin{aligned} \theta(p+1, q+1, r, k, l) - \theta(p, q, r, k, l) &= -\frac{2\lambda}{(p+2)(p+3)(p+4)} \\ &\quad - \frac{2\lambda}{(q+2)(q+3)(q+4)} - b_2 \end{aligned}$$

Hence, $b_2 = 0$. □

The following two propositions are proved in the same way as Proposition 3.12.

Proposition 3.14. *There exists $b_3 \in \mathbb{Q}$ such that for any $p, q, r \geq 3$ we have*

$$\eta(p, q, r) = \frac{\lambda}{(p+2)(p+3)} - \frac{\lambda}{(q+2)(q+3)} + \frac{\lambda}{(r+2)(r+3)} + b_3$$

Proposition 3.15. *There exists $b_4 \in \mathbb{Q}$ such that for any $p, q, r, k \geq 3$ we have*

$$\zeta(p, q, r, k) = \frac{\lambda}{(p+2)(p+3)} - \frac{\lambda}{(q+2)(q+3)} - \frac{\lambda}{(r+2)(r+3)} + \frac{\lambda}{(k+2)(k+3)} + b_4$$

Consider $\bar{\alpha}_6(L, x, y, z, u, v) \in \mathcal{S}_6(p, q, r, k, l)$, $p, q, r, k, l \geq 2$. Let Δ be the triangle that contains the edge $\{x, y\}$ and does not coincide with the triangle $\{x, y, z\}$. Let $L^{(1)}$ be the PL sphere obtained from L by the bistellar move associated with Δ . Arguing as in the proof of Proposition 3.13, we see that

$$c(\bar{\alpha}_6(L, x, y, z, u, v)) - c(\bar{\alpha}_6(L^{(1)}, x, y, z, u, v)) = \frac{2\lambda}{(p+2)(p+3)(p+4)} + \frac{2\lambda}{(q+2)(q+3)(q+4)}$$

Consequently for any $p, q, r, k, l \geq 2$ the function c is constant on $\mathcal{S}_6(p, q, r, k, l)$ and the formula of Proposition 3.12 holds. Similarly the formulae of Propositions 3.14 and 3.15 hold for any $p, q, r \geq 2$ and $p, q, r, k \geq 2$ respectively. The formula of Proposition 3.14 holds also for $p = q = r = 1$.

Obviously, $\eta(1, 1, 1) = 0$ and $\zeta(2, 2, 2, 2) = 0$. Hence $b_3 = -\frac{\lambda}{12}$ and $b_4 = 0$. It is easy to prove that $\theta(2, 2, 2, 2, 2) = -5\eta(2, 2, 2) = \frac{\lambda}{6}$. Hence, $b_5 = -\frac{\lambda}{12}$.

Consequently $c(\bar{\alpha}) = \lambda c_0(\bar{\alpha})$ for any generator $\bar{\alpha} \in \mathcal{S}$.

3.5 The constant λ

We have $s^*(\phi) = \lambda c_0$ for some $\lambda \in \mathbb{Q}$. To prove that $\lambda = 1$ we need to check that the formula of the Corollary 3.2 holds for some oriented 4-dimensional combinatorial manifold with nonzero first Pontrjagin number.

In [11] W. Kühnel and T. F. Banchoff constructed the triangulation $\mathbb{C}P^2$ with 9 vertices (see also [12]). The links of all vertices of this triangulation are isomorphic to the same 3-dimensional oriented PL sphere L . (L is one of the two 3-dimensional PL spheres that are not polytopal spheres, see [9]). One can number the vertices of L so that the 3 dimensional simplices are given by:

1243	3476	5386	7165
1237	3465	4285	1785
1276	4576	4875	1586
2354	2385	4817	1682
2376	2368	4371	1284

Consider the following sequence of 9 bistellar moves. This sequence transforms L to the boundary of 4-dimensional simplex.

- 1) Replace the simplices 1243, 1237, and 4371 with the simplices 1247 and 3274.
- 2) Replace the simplices 2354, 2385, and 4285 with the simplices 2384 and 3584.
- 3) Replace the simplices 7165, 1785, and 1586 with the simplices 1786 and 5687.
- 4) Replace the simplices 1786, 1682, and 1276 with the simplices 1278 and 6287.
- 5) Replace the simplices 1247, 1278, 1284, and 4817 with the simplex 2487.
- 6) Replace the simplices 3274, 2384, and 2487 with the simplices 2387 and 3487.

- 7) Replace the simplices 2387, 6287, 2376, and 2368 with the simplex 6387.
- 8) Replace the simplices 3465, 5386, and 3584 with the simplices 4386 and 5486.
- 9) Replace the simplices 4386, 6387, 3476, and 3487 with the simplex 6487.

Using this sequence of bistellar moves the formula of Corollary 3.2 can be checked by a direct calculation.

3.6 Denominators of local formulae's values

Denote by $\mathcal{T}_{n,l}$ the set of all oriented $(n-1)$ -dimensional PL spheres with not more than l vertices. Let $f \in \mathcal{T}^n(\mathbb{Q})$ be a local formula. By $\text{den}_l(f)$ denote the least common multiple of the denominators of the values $f(L)$, where L runs over all elements of $\mathcal{T}_{n,l}$.

Proposition 3.16. *Let $\psi \in H^n(\mathcal{T}^*(\mathbb{Q}))$ be an arbitrary cohomology class. Then there exists a local formula f representing the class ψ and the integer constant $C \neq 0$ such that the number $\text{den}_l(f)$ is a divisor of $C(l+1)!$ for any $l \geq n$.*

Proposition 3.17. *Let f be an arbitrary local formula representing the generator ϕ of the group $H^4(\mathcal{T}^*(\mathbb{Q}))$. Then the number $\text{den}_l(f)$ is divisible by the least common multiple of the numbers $1, 2, 3, \dots, l-3$ for any even $l \geq 10$.*

The proof of Proposition 3.16 uses Theorem 2.1 and results of [13]. Proposition 3.17 is a corollary of Theorem 3.1.

Corollary 3.3. $H^4(\mathcal{T}^*(G)) = 0$ for any subgroup $G \subsetneq \mathbb{Q}$.

I am grateful to V. M. Buchstaber for statements of the problems and for constant attention to my work. I am grateful to L. A. Alania, I. V. Baskakov, I. A. Dynnikov, M. È. Kazarian, and A. S. Mischenko for usefull discussions.

References

- [1] Alexander J. W., *The combinatorial theory of complexes*, Ann. Math. **31** (1930), no. 2, 292–320.
- [2] Buchstaber V. M., Panov T. E., *Torus Actions and Their Applications in Topology and Combinatorics*, University Lecture Series **24** (2002), Amer. Math. Soc., R. I.: Providence.
- [3] Cheeger J., *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. **18** (1983), no. 4, 575–657.
- [4] Gabriellov A. M., Gelfand I. M., Losik M. V., *Combinatorial calculation of characteristic classes* (Russian), Funktsional Anal. i Prilozhen. **9** (1975), no. 2, 12–28, no. 3, 5–26.
- [5] Gabriellov A. M., Gelfand I. M., Losik M. V., *Local combinatorial formula for the first Pontrjagin class* (Russian), Funktsional Anal. i Prilozhen. **10** (1976), no. 1, 14–17.
- [6] Gaifullin A. A., *On local formulae for combinatorial Pontrjagin classes of manifolds* (Russian), Uspekhi Mat. Nauk **59**(2004), no. 2, 189–190.

- [7] Gaifullin A. A., *Local formulae for combinatorial Pontrjagin classes* (Russian), Izv. Ross. Akad. Nauk, to appear.
- [8] Gelfand I. M. and MacPherson R. D., *A combinatorial formula for the Pontrjagin classes*, Bull. Amer. Math. Soc. **26** (1992), no. 2, 304–309.
- [9] Grünbaum B., Sreedharan V. P., *An enumeration of simplicial 4-polytopes with 8 vertices*, J. Combin. Theory, Ser. A **2** (1967), 437–465.
- [10] Kazarian M. È., *The Chern-Euler Number of Circle Bundle via Singularity Theory*, Mathematica Scandinavica **82** (1998), 207–236.
- [11] Kühnel W., Banchoff T. F., *The 9-vertex complex projective plane*, Math. Intell. **5** (1983), no. 3, 11–22.
- [12] Kühnel W., Lassmann G., *The unique 3-neighborly 4-manifold with few vertices*, J. Combin. Theory, Ser. A **35** (1983), no. 2, 173–184.
- [13] Levitt N., Rourke C., *The existence of combinatorial formulae for characteristic classes*, Trans. Amer. Math. Soc. **239** (1978), 391–397.
- [14] Pachner U. *P.L. homeomorphic manifolds are equivalent by elementary shellings*, European J. Combin. **12** (1991), no. 2, 129–145.
- [15] C. P. Rourke, B. J. Sanderson, *Introduction to Piecewise-Linear Topology*, Springer-Verlag, Berlin, 1982.
- [16] Steinitz E. and Rademacher H., *Vorlesungen über die Theorie der Polyeder*, 1934.